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A RATIONAL APPROACH ON THE STUDY OF A UNIAXIAL BUCKLING MODEL

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Abstract. *The nature of buckling and the quantitative calculation of different parameters which are involved on the load carrying capacity of columns is the aim of this project. Current design methodologies for columns in compression or a combined action of it with bending, assume failure occurs when the column buckles about the minor axis of its cross section, without paying considerable attention on any imperfections, which may arise either from out of the column's straightness or from deflections due to the bending moments developed on the joints, either the columns are considered as independent members or as a part of a rigid jointed steel frame. The target of this project is to develop and investigate a theoretical model, where all sources of imperfections are involved, so that this model could be used at a subsequent stage to investigate the biaxial buckling. A number of relevant concepts and calculations is built along with a systematic study of all kinds of possible imperfections concerning the uniaxial buckling, with respect to an upper and lower bound to the collapse load..*

1 INTRODUCTION

The behavior of structural members in compression and bending is a major area of structural engineering research world wide. The aim of this research has always been to enable engineers to design safe and efficient structures and to achieve a better understanding of the underlying principles which governs the behavior of such members.

1.1 Methods of structural analysis

Three main categories of structural analysis methods have been used by engineers to analyse struts and indeed other types of structural members.

a. *Linear elastic analysis*, where a linear relationship is assumed to exist between the load and displacements everywhere in the structure. This method provides accurate predictions of behavior of structures not subjected to destabilizing effects related to either member or overall structure instability.

b. *Non linear elastic analysis*, when the effects of member or overall structure instability are to be taken into account. The relationship between load and displacements is no longer linear.

c. *Plastic analysis*, where, an analysis of the configuration of the structure or the stress distribution in a section at failure is carried out and used to predict the load carrying capacity of the structure or the member under consideration.

1.2 Historical background

Van Musschenbroek [1] published the first paper concerned with the strength of axially loaded members in 1729. He observed from experiments conducted by himself that the strength of a "long" axially loaded strut is inversely proportional to the square of its length. No analytical relationship was established at the time.

Euler [2] published his paper in 1759 in which he presented an analysis of a concentrically loaded, perfectly straight elastic strut. His analysis was based on the assumption that an originally straight column would remain straight until the load reaches a certain critical value after which deformations take place and the column becomes unstable. An analysis of the structure in its slightly deformed configuration, based on the solution to Bernoulli's equation

$$-EI\frac{d^2w}{dx^2} = M \tag{1}$$

gives the value of the critical load as

$$P_c = \frac{\pi^2 EI}{KL^2} \tag{2}$$

where P_c is the Euler critical load and K is a constant which depends on the boundary conditions.

In 1807 Young [3] stated that an initially bent strut will experience lateral deformation from the onset of loading regardless of how small the load might be. He concluded that irregularities that appear in experiments could be attributed to the presence of unavoidable initial out of straightness, material non homogeneity and loading eccentricity. His work constitute the first acknowledgement of the effects various imperfections have on the strength of columns.

The French engineer Considère [4] presented in 1889 the 'Reduced Modulus Theory' in which he analyzed the column on the assumption that it remains straight until the stress exceeds the limit of proportionality, after which the column starts to bend. He reasoned that the resulting curvature would reduce the strain in the concave side of the column and increase it in the convex side. He then suggested that if the stress in the column exceeds the limit of proportionality, the elastic modulus E in the Euler equation should be replaced with a reduced modulus E_r having a value somewhere between the elastic modulus and the tangent modulus E_t, i.e.

$$E \le E_r \le E_t \tag{3}$$

In the same year 1889, Engesser [5] presented a similar line of thought when he proposed a Tangent Modulus Theory based on the assumption that, the stress in the column can reach a certain limit before the onset of lateral deformations, and that when this is the case, the elastic modulus should be replaced with the tangent modulus.

Both of the reduced modulus theory and the tangent modulus theory are examples of non linear elastic analysis whilst the Euler theory is based on linear elastic analysis. Furthermore, all three theories fail to acknowledge the effects of imperfections on the stability of struts.

In 1886 Professor Ayrton [6] presented a joint paper with Perry in which they published the results of their investigation on the effects of imperfections on column buckling. Their work is regarded by many as the most rational approach to the analysis of real columns behavior. They assumed that the total deflection ξ in a strut with pinned ends, can be represented by an Equivalent Initial Imperfection ξ_0, having a sine form

$$\xi(x) = \xi_0 \sin\left(\frac{\pi x}{L}\right). \tag{4}$$

Using this initial bow, they were able to derive an analytic relationship between the buckling load P_b, the Euler critical load P_E and the equivalent initial deflection. The Ayrton – Perry equation is

$$(P_y - P_b)(P_E - P_b) = P_b P_E \rho \ , \tag{5}$$

where ρ is a non dimensional parameter given by

$$\rho = \xi_0 \frac{A}{Z}. \tag{6}$$

They were also able to relate the total equivalent imperfection to central deflection of the column by the equation

$$\frac{1}{\delta} = \frac{1}{P} \cdot \frac{P_E}{\xi_0} - \frac{1}{\xi_0} \ , \tag{7}$$

where δ is the measured central deflection.

Eq. (5) enabled the imperfection effects to be quantified explicitly for the first time whilst (7) enabled the direct measurement of the imperfection parameter from laboratory experiments. The Ayrton – Perry equation is the basis of the column design curves in BS 449 and BS 5950.

2 NOTATION

- E Young's modulus.
- M Bending moment.
- P Applied load.
- P_E Euler critical load.
- P_{ci} Euler critical load at the ith mode.
- P_y Squash load.
- P_{fh} Load at first hinge.

P_b Buckling load.
P_{fy} Load corresponding to the first yield.
ρ Imperfection parameter.
ξ_o Total equivalent imperfection.
A Cross sectional area.
Z Plastic modulus of the section.
I Second moment of area.
c Spring constant.
i indicate the mode i.e 1,2,3...
Φ Characteristic shape function.
L_e Effective length.
w Central deflection
w_i Maximum deflection at the i^{th} mode.
w_o Amplitude factor associated with geometrical imperfections
w^p Amplitude factor associated with proportional loading imperfections.
w^n Amplitude factor associated with non proportional loading imperfections.
σ_m Maximum stress on the cross section.
m^p Moment induced by the proportional loading.
m^n Moment induced by the non-proportional loading.
α Curvature function.

3 THEORETICAL BACKGROUND

In this part the Elastic critical analysis and the Imperfection approach will be explored in some detail. These two theories have been chosen because of their relevance to the theoretical model that will be developed in the coming paragraphs.

3.1 Elastic critical analysis of struts. Eigenvalue & Eigenvector problem

A perfectly straight column, restrained at both ends, with no want of homogeneity or locked stresses is shown in Fig. 1. The column is acted upon by a load P applied at the centroid of its section. As a result of this load, the column is deflected as shown in the figure.

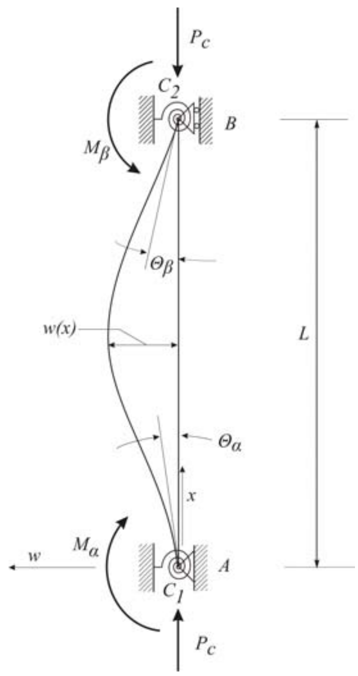

Figure 1: A perfectly straight column analysis.

The differential equation of equilibrium and compatibility of the column is

$$EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} = 0 \ . \quad (8)$$

Eq. (8) is a homogeneous linear differential equation with a general solution

$$w(x) = A_1 \sin(kx) + A_2 \cos(kx) + A_3 x + A_4 \ , \tag{9}$$

$$\text{where} \quad k = \sqrt{\frac{P}{EI}} \ . \tag{10}$$

The constants A_1 to A_4 can be calculated from the four boundary conditions related to the support conditions at both ends of the column. The mathematical formulation of the boundary conditions yields a set of four simultaneous linear equations in the four constants. These equations, in a matrix notation take the form

$$[f(k)]\cdot[\boldsymbol{A}] = 0 \ . \tag{11}$$

For an arbitrary value of the load P_{ci}, i.e. $\{k_i\}$, Eq. (11) is satisfied only when $A_1 = A_2 = A_3 = A_4 = 0$, indicating that the column remains straight. If a deflected configuration of the column is to be found, then the determinant of the coefficient matrix $\Delta[f(k)]$ must vanish. This is an Eigenvalue problem which gives an infinite number of solutions, $k_1, k_2, \ldots k_\infty$, corresponding to an infinite number of critical loads $P_{c1}, P_{c2}, \ldots P_{c\infty}$, where

$$P_{ci} = EI k_i^2 \ . \tag{12}$$

The constants A_1 to A_4 can now be obtained by substituting k_i in Eq. (11). Since the coefficients matrix is now singular, an explicit solution for the constants A_1 to A_4 is not possible and only the Eigenvectors can be obtained. This implies that three of the constants can be expressed in terms of the fourth, and the general solution to Eq. (8) is

$$w_i(x) = \overline{w}_i \left[\sin(k_i x) + A_i \cos(k_i x) + B_i x + C_i\right], \tag{13}$$

where $\overline{w}_i$ is an arbitrary constant and A_i, B_i, C_i are known values.

3.1.1 Solution of the Eigenvalue problem

For columns with simple end conditions, an analytic solution of the Eigenvalue problem can be obtained. The critical load and the deflected shape of a column with both ends pinned, i.e. for $k_i = \frac{i\pi}{L}$, will be:

$$P_{ci} = \frac{(i\pi)^2 EI}{L^2} \quad \text{and} \quad w_i(x) = w_i \sin\left(\frac{i\pi x}{L}\right) \tag{14}$$

Similarly, a column with fixed ends, for $k_i = \frac{2i\pi}{L}$, will have the solution:

$$P_{ci} = \frac{(2i\pi)^2 EI}{L^2} \quad \text{and} \quad w_i(x) = \frac{w_i}{2}\left[1 - \cos\left(\frac{2i\pi x}{L}\right)\right] \tag{15}$$

Analytic solutions are not always feasible for columns with more complicated end conditions. For such columns numerical solutions are more appropriate.

As an illustrative example, let us consider the column shown in Fig. 1, assuming that both ends have rotational restraints with spring constants c_1 and c_2. Eq. (11) becomes

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \sin(kL) & \cos(kL) & L & 1 \\ c_1 k & k^2 EI & c_1 & 0 \\ k^2 EI \sin(kL) + c_2 k \cos(kL) & k^2 EI \cos(kL) - c_2 k \sin(kL) & c_2 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0 \ . \tag{16}$$

The Eigenvalues of Eq. (16) can be obtained by a computer routine that searches for successive values of k at which the determinant $\Delta[f(k)]$ vanishes. The program is based on the following simple algorithm:

a. Initially a small value of $k = k_o$ is assumed.
b. The determinant $\Delta[f(k)]$ is then evaluated.
c. If the absolute value of the determinant is less than the required accuracy, then k_0 is an Eigenvalue to the determinant; if not continue at step (d) below

d. Increment of the value k to a $k_1 = k_0 + \delta k$, where δk is a very small given increment, and new evaluation of the determinant.

e. If the value of the determinant changes its sign, then put $k_1' = k_o - \delta k'$ where $\delta k' = \delta k/2$ and repeat steps (c) through to (e) until convergence to zero is achieved.

f. If no change of sign is observed in (e) then put $k = k_1$ and repeat steps (c) to (e) until a solution is obtained.

g. Start the routine at step (a) with an initial value of k slightly greater than the previous solution to obtain the next Eigenvalue.

3.1.2 Mode shapes & characteristic functions

Eq. (13) is indeterminate as to magnitude because $\overline{w_i}$ is still an arbitrary constant, but definite as to shape, since A_i, B_i, C_i are known. It can be written as

$$w_i(x) = \overline{w_i}\Phi_i(x)\ , \tag{17}$$

where the function

$$\Phi_i(x) = \sin(k_i x) + A_i \cos(k_i x) + B_i x + C_i \tag{18}$$

is the so called characteristic shape function of the ith critical mode.

The above critical modes can be calculated directly from Eq. (11)

3.1.3 Properties of mode shapes

3.1.3.1 Harman's principle

It has been, shown [7] that the buckled shape of a column with any boundary conditions is a sine curve, provided that an appropriate set of Cartesian coordinates is employed (Harman's principle).

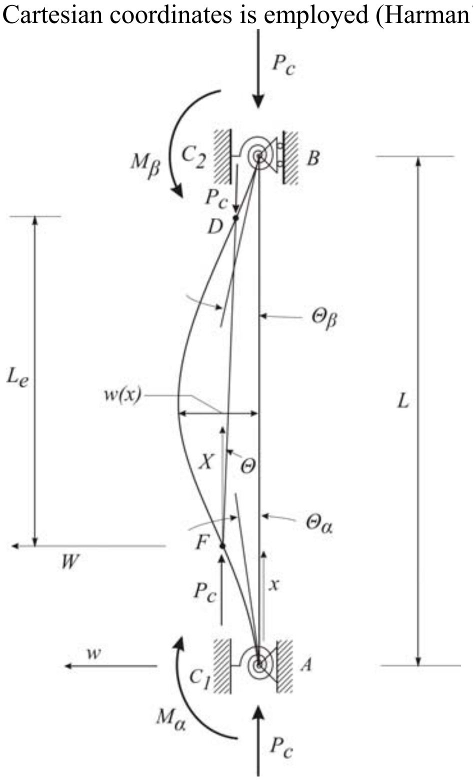

Figure 2: The concept of Effective Length

In the column of Fig. 2, points F and D are points of contraflexure and the resultant moment is zero. If we choose the X axis to pass through F and D with the origin at F, the segment F-D is essentially a column with pinned ends of length L_e. If the angle Θ is small, then the load on the shorter column F-D is equal to P_c. By correlating the elastic critical load of the column in Fig. 2 with the Euler load of a pin-ended column, we can consider the column in the figure as simply supported, having a length L_e. This result asserts a physical meaning of the effective length, as the distance between two adjacent points of contraflexure. The critical load can be written as

$$P_{ci} = \frac{\pi^2 EI}{L_{ei}^2} \quad . \tag{19}$$

3.1.3.2 Modal shapes for columns with symmetrical end conditions

Symmetry of end conditions can be assumed for columns in intermediate floors in multi-story framed structures and for any controlled experimental testing where identical end supports, clamps or spring supports are provided. This case is of particular importance to this project since the experimental model adopted had symmetrical end conditions.

Fig. 3 shows the first three critical mode shapes of a column with symmetrical end conditions.

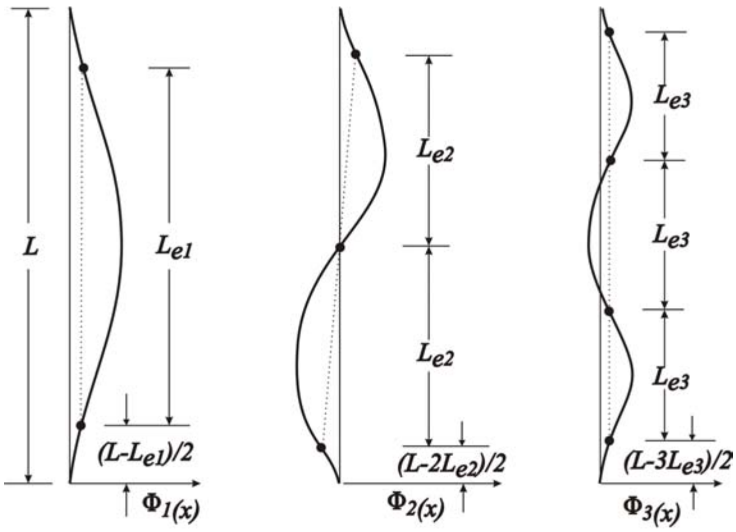

Figure 3: Symmetric Column Eigenvectors

From Harman's Principle, the buckled shape of the column is a sine curve, $\overline{W} = A_0 \sin(\gamma X)$, where the distance between two points of contraflexure is the effective length L_{ei}. This gives a value for $\gamma = \pi / L_{ei}$. The general solution to the equilibrium equation in terms of the local coordinates (X,W), corresponding to Eq. (13) is

$$W(X) = \overline{A}\left[\sin(\gamma X) + B\cos(\gamma X) + CX + D\right] \quad . \tag{20}$$

If we express this equation in terms of the global system (x,w) and substitute for γ, noting that $X = x - \frac{L - iL_{ei}}{2}$, we get

$$w = \overline{A}\left[\sin\left(\frac{\pi}{L_{ei}}\left[x - \frac{L - iL_{ei}}{2}\right]\right) + B\cos\left(\frac{\pi}{L_{ei}}\left[x - \frac{L - iL_{ei}}{2}\right]\right) + Cx + \overline{D}\right] = \overline{A}\Phi_i \quad , \tag{21}$$

where

$$\Phi_i = \sin\left(\frac{\pi x}{L_{ei}} + \frac{i\pi}{2} - \frac{\pi L}{2L_{ei}}\right) + B\cos\left(\frac{\pi x}{L_{ei}} + \frac{i\pi}{2} - \frac{\pi L}{2L_{ei}}\right) + Cx + \overline{D} \quad . \tag{22}$$

The constants B, C and $\overline{D}$ can be calculated from the conditions

$\Phi_i = 0$ for $x = 0$ and $x = \mathrm{L}$

$\frac{d^2\Phi_i}{dx^2} = 0$ for $x = \frac{L - iL_{ei}}{2}$.

Applying these boundary conditions yields

$$\Phi_i = \sin\left(\frac{\pi x}{L_{ei}} + \frac{i\pi}{2} - \frac{\pi L}{2L_{ei}}\right) - \frac{2x}{L}\sin\left(\frac{\pi L}{2L_{ei}}\right)\cos\left(\frac{i\pi}{2}\right) - \sin\left(\frac{i\pi}{2} - \frac{\pi L}{2L_{ei}}\right) \quad . \tag{23}$$

Eq. (23) enables the bending moment at any point x to be calculated when the column has buckled elastically in its i^{th} mode, since

$$M = -EI\frac{d^2w(x)}{dx^2} = -EI\sum_{i=1}^{\infty} w_i \Phi_i''(x) = \sum_{i=1}^{\infty} P_{ci} w_i \sin\left(\frac{\pi x}{L_{ei}} + \frac{i\pi}{2} - \frac{\pi L}{2L_{ei}}\right), \tag{24}$$

where P_{ci} and w_i are the critical load and amplitude factor of mode i.

Eq. (23) has three terms, the first is a sine curve, the second term represents a rotation from one system of coordinate to the other, and the third term a translation between the two abscissas.

3.1.3.3 Orthognality of mode shapes

The mode shapes of section 3.1.3. satisfy the so called orthognality relations

$$\int_0^L \Phi_i'(x) \cdot \Phi_j'(x) dx = 0 \text{ and} \tag{25}$$

$$\int_0^L EI\Phi_i''(x) \cdot \Phi_j''(x) dx = 0, \tag{26}$$

where Φ_i and Φ_j are two different characteristic functions associated with two different critical loads. A further relation exists for each mode and is given by

$$P_{ci}\int_0^L \left[\Phi_i'(x)\right]^2 dx = \int_0^L EI\left[\Phi_i''(x)\right]^2 dx \quad . \tag{27}$$

It can be shown [8] that Φ_i and Φ_j form a complete set of orthogonal functions. The deflected shape of a column can then be written as a converging series,

$$w(x) = \sum_{i=1}^{\infty} w_i \Phi_i(x), \tag{28}$$

where w_i is an amplitude factor associated with mode i .

3.2 Imperfection analysis

3.2.1 Sources of imperfections

In practical situations, perfect columns never exist. The presence of initial out of straightness, material non homogeneity, residual stresses, deflection arising from loading of other parts of the structure or eccentric loading, should be expected and allowed for by the designer. In addition ,the manufacturing process of hot rolled sections, often leads to stresses being locked in the section due to non uniform rate of cooling.

The combined effect of all the factors mentioned above is referred to by the term *imperfections*. In general, Imperfections can be classified into two main categories, *geometric* imperfections, and *loading* imperfections. Loading imperfection can further be classified into *proportional* and *non proportional* loading imperfections. Geometric imperfections encompass the effects of out of straightness, material defects and residual stresses. Proportional loading, refers to loads which are proportional to the axial load on the column whilst a non proportional loading is independent of the axial load.

3.2.2 Geometric imperfections

Fig. 4 shows a column with initial deflection $w^o(x)$. An axial load P is acting on the column. As a result of the application of the axial load, the deflection is increased by an amount $w(x)$.

The imperfection can be expressed in terms of a converging series of the characteristic functions $\Phi_i(x)$ as

$$w^o(x) = \sum_{i=1}^{\infty} w_i^o \Phi_i(x), \tag{29}$$

where w_i^o is an unknown amplitude factor termed the modal geometric imperfection.

It should be noted, that in the original Ayrton – Perry equation, the imperfections were assumed to take the form of a half-sine curve. This is the shape of the characteristic function for a column with pinned ends, buckling in its first mode (Eq. 14). In 1932, Southwell [9], showed that for a column with any arbitrary end conditions, the first mode of buckling has the principal contribution to the imperfection function (29). The differential equation of equilibrium (8) for the column in Fig. 4 is

$$EI\left[w^{t^{iv}}(x) - w^{o^{iv}}(x)\right] + Pw^{t^{''}}(x) = 0\,. \tag{30}$$

Substituting for $w^o(x)$ and $w(x)$ from Eqs (29) and (28) respectively into (30),

$$EI\left[\sum_{i=1}^{\infty} w_i^t \Phi_i^{iv}(x) - \sum_{i=1}^{\infty} w_i^o \Phi_i^{iv}(x)\right] + P\sum_{i=1}^{\infty} w_i^t \Phi_i^{''}(x) = 0\,. \tag{31}$$

Since P_{ci} and $\Phi_i(x)$ satisfy the Euler Eq. (8), we can write

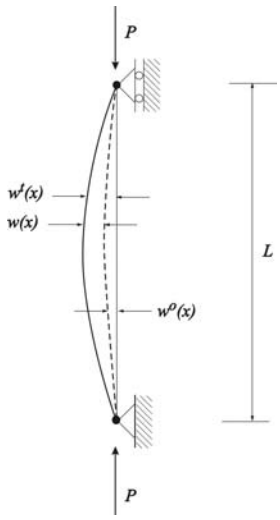

Figure 4: Geometrical Imperfections

$$EI\Phi_i^{iv}(x) + P_{ci}\Phi_i^{''} = 0\,. \tag{32}$$

Substituting $\Phi_i^{iv}(x)$ from (32) into (31),

$$-\sum_{i=1}^{\infty} P_{ci} w_i^t \Phi_i^{''}(x) + \sum_{i=1}^{\infty} P_{ci} w_i^o \Phi_i^{''}(x) + P\sum_{i=1}^{\infty} w_i^t \Phi_i^{''}(x) = 0\,. \tag{33}$$

Multiplying Eq. (28) by $\Phi_j(x)$, integrating over the whole length, and using the orthgonality relations of paragraph 3.1.3.3 we can prove that Eq. (33) holds for each mode separately, i.e

$$w_i^t = \frac{P_{ci}}{P_{ci} - P} \cdot w_i^o\,, \tag{34}$$

where w_i^t is the amplitude factor of the *total* deflection associated with mode *i.*

The amplitude factor for non linear deflection w_i *due to the axial load* is given by

$$w_i = \frac{P}{P_{ci} - P} \cdot w_i^o\,. \tag{35}$$

3.2.3 Proportional loading imperfections

The general case of proportional loading imperfection, is that, of a beam column acted upon by a lateral load and subjected to end moments M_a and M_b. It is assumed here that all loads are proportional to the axial load on the column. The column is shown in Fig. 5.

If $w^p(x)$ is the deflection due to proportional loading (without the contribution of axial load), it can be expressed it in terms of an infinite series of the characteristic functions, as

$$w^p(x) = \sum_{i=1}^{\infty} w_i^p \Phi_i(x)\,. \tag{36}$$

The differential equation of equilibrium prior to the application of axial load is

$$EI(w^{p})^{iv} = q(x)\,. \tag{37}$$

After the application of axial load, the column takes its final configuration. The differential equation of equilibrium becomes

$$EIw^{t^{iv}}(x) + Pw^{t''}(x) = q(x)\,. \tag{38}$$

Combining Eq. (36) and (37), results in

$$EI[(w^{t})^{iv} - (w^{p})^{iv}] + P(w^{t})'' = 0\,. \tag{39}$$

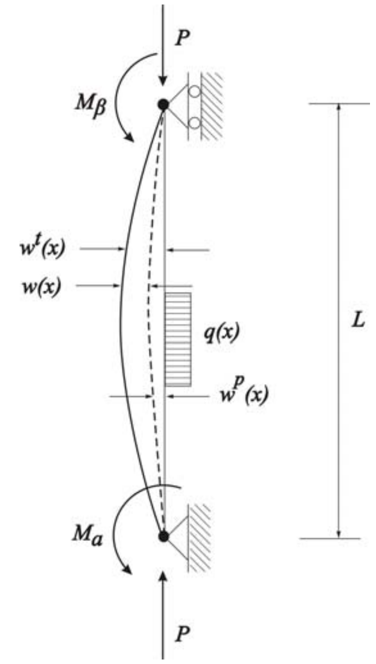

Figure 5: Loading Imperfections.

Eq. (39) is identical to the equilibrium differential Eq. (30) for the geometrical imperfections, with w^p substituted for w^o. It follows that the relations between the total deflection, the deflection due to the axial load and the non proportional loading imperfections are similar to those in section (3.2.2), namely

$$w_i^t = \frac{P_{ci}}{P_{ci} - P} \cdot w_i^o \text{ and } w_i = \frac{P}{P_{ci} - P} \cdot w_i^o\,. \tag{40}$$

Similar equations can be derived for the case of non-proportional loading imperfections w^n. It can be seen that the equations relating the amplitude factors of imperfections arising from loading and geometry are identical. It follows that, when the three classes of imperfections, mentioned in section (3.2.1), are present, a generalized equation can be formulated as

$$w_i^t = \frac{P_{ci}}{P_{ci} - P} \cdot \xi_i \quad \text{and} \quad w_i = \frac{P}{P_{ci} - P} \cdot \xi_i\,, \tag{41}$$

$$\text{where} \quad \xi_i = w_i^o + w_i^p + w_i^n \tag{42}$$

The total deflection of the column at any point becomes

$$w_i^t(x) = \sum_{i=1}^{\infty} \frac{P_{ci}}{P_{ci} - P} \cdot \xi_i \Phi_i(x) \tag{43}$$

and the non-linear deflection due to the application of axial load is

$$w(x) = \sum_{i=1}^{\infty} \frac{P}{P_{ci} - P} \cdot \xi_i \Phi_i(x) . \quad (44)$$

Substituting for *w(x)* from (41) into (24), we obtain an expression for the bending moment in the column as

$$M = -EI \sum_{i=1}^{\infty} \frac{P}{P_{ci} - P} \cdot \xi_i \Phi_i''(x) + m^p(x) + m^n(x) , \quad (45)$$

where $m^p(x) = -EIw^{p''}(x)$ and $m^n(x) = -EIw^{n''}(x)$
are linear moments obtained by linear elastic analysis of the whole structure of which the column is a part. It is assumed here that the geometrical imperfections are stress free. The parameter ξ_i is a representation of the *Total Equivalent Imperfection***.** The concept of a total equivalent imperfection parameter that accounts for imperfections resulting from *ALL* sources was first developed at University College London in 1993.

4 CONCLUSIONS

In this paper which is the first part of a research dealing with the theoretical investigation of a rational approach to the study of Biaxial Buckling Model, the conclusions that can be derived are:

- The imperfection approach to the column design buckling is the only rational method that can be used to analyze and design real columns.
- Availability of cheap, fast and efficient computer programs suggest alternative approaches to column design, with less emphasis on empirical formulae and more emphasis on rational thinking.

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